Minimal Effort in Micro-Founded Tullock Contests*

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Abstract

In a micro-founded Tullock contest, derived from the rank-order tournament framework of Lazear and Rosen (1981), the minimal effort exerted by players should be strictly positive—specifically, one rather than zero. This condition guarantees equilibrium existence in the basic model, and extends to various generalizations of the Tullock framework.

1 Introduction

Tullock contests, initiated in the seminal works of Tullock (1975, 1980), describe a simple one-stage n-player game (with $n \geq 2$), in which each player i may exert a non-negative effort $x_i \geq 0$ at a cost of x_i, to compete for a prize of value $r > 0$. The probability that player i wins the prize is given by

$$p_i(x_i, x_{-i}) = \frac{x_i}{\sum_j x_j}, \tag{1}$$

provided that $\sum_j x_j > 0$; otherwise, the probability is either $1/n$ or zero.[1] The functions p_i are commonly referred to as *contest success functions*. These types of contests have been extensively studied, particularly in relation to various forms and extensions of the contest success functions.

Lazear and Rosen (1981) propose a similar contest structure, but with a different class of success functions: each player i is evaluated according to $x_i + \epsilon_i$, where the ϵ_i are i.i.d. non-atomic random variables (errors) with sufficiently wide support. In this setting, player i exerts effort x_i at a similar cost and wins the prize $r > 0$ with probability $\Pr(x_i + \epsilon_i > x_j + \epsilon_j,\ \forall j \neq i)$. These are generally known as *rank-order tournaments*.

Crucially, one can select appropriate distributions for the error terms to approximate the Tullock success function, subject to a minor adjustment: the minimal effort level must be 1 rather than 0. This modification eliminates the discontinuity of $p_i(\cdot)$ at the origin and ensures the existence of a Nash equilibrium via standard fixed-point arguments, as in Nash (1950).

Although Tullock did not provide a micro-foundation for the proposed success functions, several subsequent studies have done so.[2] For instance, Skaperdas (1996) presents an axiomatic derivation of contest success functions—later extended by Clark and Riis, 1998; Jia (2008) derives such functions based on rank-order tournaments with multiplicative error terms; Matějka and McKay (2015)

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[1] As specified later, costs need not be linear.

[2] See Konrad (2007) and Jia et al. (2013) for comprehensive reviews.

employs the rational inattention framework (initiated by Sims, 2003) and discrete choice models akin to Luce (1959).

For our purposes—deriving Tullock's success function from the rank-order tournaments of Lazear and Rosen (1981)—we rely on the result of McFadden (1974), particularly Lemma 1 therein. We now provide a brief review of this result and its key implications.

2 From rank-order tournaments to Tullock's success functions

Consider the previously defined rank-order tournament and assume that all ϵ_i are independent and identically distributed according to the extreme-value distribution, such that $\Pr(\epsilon \le k) = F_\epsilon(k) = \exp(-e^{-k})$ for every $k \in \mathbb{R}$. Then,

$$\begin{aligned}
\Pr(x_i + \epsilon_i > x_j + \epsilon_j,\ \forall j \neq i) &= \int_{-\infty}^{\infty} f_{\epsilon_i}(k) \prod_{j \neq i} F_{\epsilon_j}(x_i - x_j + k)\, dk \\
&= \int_{-\infty}^{\infty} e^{-k} \exp(-e^{-k}) \cdot \prod_{j \neq i} \exp(-e^{x_j - x_i - k})\, dk \\
&= \int_{-\infty}^{\infty} e^{-k} \exp(-e^{-k}) \cdot \exp\left(-\sum_{j \neq i} e^{x_j - x_i - k}\right) dk.
\end{aligned}$$

Substituting $t = e^{-k}$, we obtain

$$\begin{aligned}
\Pr(x_i + \epsilon_i > x_j + \epsilon_j,\ \forall j \neq i) &= \int_0^{\infty} \exp(-t) \cdot \exp\left(-t \sum_{j \neq i} e^{x_j - x_i}\right) dt \\
&= \int_0^{\infty} \exp\left(-t \left[1 + \sum_{j \neq i} e^{x_j - x_i}\right]\right) dt \\
&= -\left.\frac{\exp\left(-t\left[1 + \sum_{j \neq i} e^{x_j - x_i}\right]\right)}{1 + \sum_{j \neq i} e^{x_j - x_i}}\right|_0^{\infty} \\
&= \frac{1}{1 + \sum_{j \neq i} e^{x_j - x_i}} = \frac{e^{x_i}}{\sum_j e^{x_j}},
\end{aligned}$$

which corresponds to the multinomial Logit model (i.e., the multinomial logistic regression classification method). Accordingly, if $x_j = 0$ for every j, the resulting success probability for each player is $\frac{1}{n}$.

We can therefore define an auxiliary game in which the action sets are $[1, \infty)$ and where each player's action is $a_i = e^{x_i}$. The utility function in the original rank-order tournament,

$$u_i(x_i, x_{-i}) = r \Pr(x_i + \epsilon_i > x_j + \epsilon_j,\ \forall j \neq i) - x_i, \quad x_i \geq 0,$$

translates in the auxiliary game to the payoff function

$$u_i(a_i, a_{-i}) = r \frac{a_i}{\sum_j a_j} - \ln(a_i), \quad a_i \geq 1.$$

Notably, in this auxiliary game, the payoff functions are continuous and the action sets are effectively compact (since sufficiently large efforts yield negative payoffs), ensuring the existence of an equilibrium by standard fixed-point arguments.

Moreover, introducing a strictly positive, strictly increasing, and sufficiently convex cost function $c(\cdot)$ can ensure that the utility functions in the auxiliary Tullock framework are concave. This guarantees the existence of a Nash equilibrium in pure strategies.

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